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ALMOST PERIODICITY, CHAIN RECURRENCE AND CHAOS

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ABSTRACT

In this paper we consider a continuous map $f: X \to X$, where X is a compact metric space. The existence of chaotic sets of f is discussed. For the special case $X = [0,1]$, we prove that f has a positive topological entropy iff it has an uncountable chaotic set in which each point is almost periodic, and iff it has an uncountable chaotic set in which each point is chain recurrent. As an application, a uniform proof for some known results will be given.

1. Introduction

Throughout this paper, X will denote a compact metric space with metric d; I is the closed interval $[0,1]$.

For a continuous map $f: X \to X$, we denote the sets of periodic points, almost periodic points, recurrent points, nonwandering points and chain recurrent points of f by $P(f)$, $A(f)$, $R(f)$, $\Omega(f)$ and $\mathrm{CR}(f)$, respectively, and the topological entropy of f by $\mathrm{ent}(f)$, whose definitions are as usual (cf. [1] where, however, "almost periodic" is called "strongly recurrent"). f^n will denote the n-fold iterate of f.

$D \subset X$ is said to be in a **chaotic set** of f, if for any different points $x, y \in D$,

$$\lim_{n\to\infty} \inf d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup d(f^n(x), f^n(y)) > 0.$$

f is said to be **chaotic**, if it has a chaotic set which is uncountable.

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For a continuous map $f\colon I \to I$, Li and Yorke [10] proved that if f has a periodic point of period 3, then it is chaotic.

Later, many sharpened results came into being in succession (see [1], [6], [7], [8], [9], [11], [13], [15], [16]). One can find in [1], [6] and [12] equivalent conditions for f to be chaotic and in [14] or [18] a chaotic map with topological entropy zero, which showed that positive topological entropy and chaos are not equivalent. On the other hand, it is known that by restricting the uncountable chaotic set to $R(f)$, or to $\overline{P(f)}$, or to $\Omega(f)$, then equivalence holds (see [3], [19], [20]).

This left us a question: Is the existence of an uncountable chaotic set of f in $A(f)$ or in $\mathrm{CR}(f)$ equivalent to $\mathrm{ent}(f) > 0$?

In the present paper, we first derive in Theorem A a sufficient condition for a map to have an uncountable chaotic set in which each point is almost periodic. We then use Theorems B and C to give a positive answer to the question.

The main results are stated as follows.

THEOREM A: *Let $f\colon X \to X$ be continuous. If f has an almost shift invariant set, then it has an uncountable chaotic set in which each point is almost periodic.*

THEOREM B: *Let $f\colon I \to I$ be continuous. If* $\mathrm{ent}(f) > 0$, *then there exists an uncountable chaotic set of f in which each point is almost periodic.*

THEOREM C: *Let $f\colon I \to I$ be continuous. If* $\mathrm{ent}(f) = 0$, *then any set containing at least two chain recurrent points of f is not chaotic*

We give the proofs of Theorems A and B in Section 2, and the proof of Theorem C in Section 3.

Theorems B and C not only give a positive answer to the above problem, but also unify the proofs of some known results. In fact, since

$$A(f) \subset R(f) \subset \overline{P(f)} \subset \Omega(f) \subset \mathrm{CR}(f)$$

(cf. [1]), we have at once

COROLLARY D: *Let $f\colon I \to I$ be continuous. Then the following are equivalent:*

(1) $\mathrm{ent}(f) > 0$.
(2) $A(f)$ *contains an uncountable chaotic set of f.*
(3) $R(f)$ *contains an uncountable chaotic set of f.*
(4) $\overline{P(f)}$ *contains an uncountable chaotic set of f.*
(5) $\Omega(f)$ *contains an uncountable chaotic set of f.*

(6) CR(f) *contains an uncountable chaotic set of* f.

Remark: In Corollary D, (1)⇔(5) was proved by Zhou [20]. (1)⇒(4) and (1)⇒(3) were given by Yang [19] and Du [3], respectively. However, (1)⇒(2) and (6)⇒(1) are new.

2. Proofs of Theorems A and B

In this section f will denote a continuous map of X into itself, Σ_k the one-sided symbol space with k symbols and σ the shift on Σ_k. For any $x \in X$, $\omega(x, f)$ denotes the set of ω-limit points of x under f.

$M \subset X$ is said to be **minimal under** f, if it is a non-void, closed and invariant subset of f and has no proper subset which is non-void, closed and invariant under f.

Definition 2.1: A compact set $\Lambda \subset X$ is said to be **almost shift invariant** if:

(1) $f(\Lambda) \subset \Lambda$.

(2) There exists for some $k \geq 2$ a continuous surjection $h: \Lambda \to \Sigma_k$ satisfying:

(a) The set $\{y \in \Sigma_k \,;\, h^{-1}(y)$ contains at least two points$\}$ is countable.

(b) $h \circ f|_\Lambda = \sigma \circ h$.

LEMMA 2.1: *For any $x \in X$ the following are equivalent:*

(1) $x \in A(f)$.

(2) $x \in A(f^n)$ *for any* $n > 0$.

(3) $x \in \omega(x, f)$ *and* $\omega(x, f)$ *is a minimal set of* f.

For a proof see [4] and [5].

LEMMA 2.2: (Lemma 3 of [2]). CR(f) =CR(f^n) *for any* $n > 0$.

LEMMA 2.3: *For any $k \geq 2$ the shift σ on Σ_k has a minimal set containing an uncountable chaotic subset.*

The proof will be given in the Appendix.

LEMMA 2.4: *Let $f: X \to X$, $g: Y \to Y$ be continuous, where X, Y are compact metric spaces. If there exists a continuous surjection $h: X \to Y$ such that $g \circ h = h \circ f$, then $h(A(f)) = A(g)$.*

Proof: By the definition of almost periodic points, we have obviously

$$h(A(f)) \subset A(g).$$

To prove the lemma, it suffices to show $h(A(f)) \supset A(g)$. For any $y \in A(g)$, $h^{-1}(\omega(y,g))$ is an invariant subset, so it contains a minimal set M of f. Clearly, $h(M) \subset \omega(y,g)$ is invariant under g. By minimality of $\omega(y,g)$, $h(M) = \omega(y,g)$. Thus there exists an almost periodic point $x \in M$ such that $h(x) = y$, which proves

$$h(A(f)) \supset A(g).$$

Proof of Theorem A: By the hypothesis in the theorem, f has an almost shift invariant set Λ, thus there is a continuous surjection $h : \Lambda \to \Sigma_k$ for some $k \geq 2$ such that for any $x \in \Lambda$,

$$h \circ f(x) = \sigma \circ h(x).$$

By Lemma 2.3, there is a minimal set $M' \subset \Sigma_k$ such that M' contains an uncountable chaotic set D' of σ. Again by Lemma 2.1, each point of M' is almost periodic. Denote, for simplicity, $g = f|_\Lambda$. By Lemma 2.4, for each $y \in D'$ we can take an $x \in A(g)$ such that $h(x) = y$. All of these points form an uncountable set of Λ, which we will denote by D. To complete the proof of the theorem, it suffices to show that D is a chaotic set of f.

For any $x_1, x_2 \in D$, there exist $y_1, y_2 \in D'$ such that $h(x_i) = y_i$ for $i = 1, 2$. First, we see easily that

$$\lim_{n\to\infty} \sup d(\sigma^n(y_1), \sigma^n(y_2)) > 0$$

implies

$$\lim_{n\to\infty} \sup d(g^n(x_1), g^n(x_2)) > 0.$$

Secondly, since Λ is an almost shift invariant set of g and D' uncountable, it follows that there exists $y_0 \in D'$ such that $h^{-1}(y_0)$ contains only one point x_0. By the chaoticity of D' and minimality of M', there exists $n_i \to \infty$ such that

$$\lim_{i\to\infty} \sigma^{n_i}(y_1) = \lim_{i\to\infty} \sigma^{n_i}(y_2) = y_0,$$

which implies

$$\lim_{i\to\infty} g^{n_i}(x_1) = \lim_{i\to\infty} g^{n_i}(x_2) = x_0.$$

Thus,

$$\lim_{n\to\infty} \inf d(g^n(x_1), g^n(x_2)) = 0.$$

Since $g = f|_\Lambda$, we see that D is a chaotic set of f.

Proof of Theorem B: Since $\text{ent}(f) > 0$, by [1] for some $N > 0$, f^N has an almost shift invariant set (cf. the proof of Prop. 15 of Chap. II in [1]). It follows from Theorem A that f^N has an uncountable chaotic set, say D, in which each point is almost periodic under f^N. Obviously, D is also a chaotic set of f. And by Lemma 2.1, $D \subset A(f)$. Hence the result follows.

3. Proof of Theorem C

LEMMA 3.1: *Let $f: X \to X$ be continuous and let $x, y \in X$. Then*

$$\lim_{n\to\infty} \inf d(f^n(x),\ f^n(y)) = 0$$

iff for any $N > 0$,

$$\lim_{n\to\infty} \inf d((f^N)^n(x),\ (f^N)^n(y)) = 0.$$

Proof: The sufficiency is obvious. We proved the necessity. Since

$$\lim_{n\to\infty} \inf d(f^n(x), f^n(y)) = 0$$

and X is compact, there exist $x_0 \in X$ and some sequence of positive integers $n_i \to \infty$ such that

$$\lim_{i\to\infty} f^{n_i}(x) = \lim_{i\to\infty} f^{n_i}(y) = x_0.$$

Let $n_i = p_i N + r_i$, where $0 \le r_i < N$. For some r with $0 \le r < N$, there exist infinitely many i's, say $i_1 < i_2 < \cdots$ such that $r_{i_1} = r_{i_2} = \cdots = r$. Set $\ell = N - r > 0$. We have

$$f^{n_{i_k}+\ell}(x) = (f^N)^{p_{i_k}+1}(x) \to f^\ell(x_0),$$
$$f^{n_{i_k}+\ell}(y) = (f^N)^{p_{i_k}+1}(y) \to f^\ell(x_0).$$

This shows

$$\lim_{n\to\infty} \inf d((f^N)^n(x), (f^N)^n(y)) = 0.$$

In the following statements, f will denote a continuous self-map of $I = [0,1]$ with entropy zero. For any $x, y \in I$, $]x, y[$ will denote the closed interval with endpoints x and y, when it is not known whether $x \le y$ or $y \le x$.

LEMMA 3.2: *If $x \in \text{CR}(f) - P(f)$, then for each $n > 0$ there exists a fixed point of f^n between x and $f^n(x)$.*

For a proof see [12].

LEMMA 3.3: *If $x \in \mathrm{CR}(f) - P(f)$, then there are no fixed points in $[a,b]$, where*

$$a = \inf\{f^{2k}(x); k = 0, 1, 2, \ldots\},$$
$$b = \sup\{f^{2k}(x); k = 0, 1, 2, \ldots\}.$$

For a proof see [1], p. 151.

LEMMA 3.4: *Suppose $x, y \in \mathrm{CR}(f)$ with $x < y$. If $[x,y] \cap P(f) \neq \emptyset$, then*

$$\lim_{n\to\infty} \inf |f^n(x) - f^n(y)| > 0.$$

Proof: To show the result, it is sufficient from Lemma 3.1 to prove that for some $N > 0$,

$$(*) \qquad \lim_{n\to\infty} \inf |(f^N)^n(x) - (f^N)^n(y)| > 0.$$

For this we divide the proof into three cases.

CASE 1: $x, y \in P(f)$. In this case, there exists an $N > 0$ such that $f^N(x) = x$ and $f^N(y) = y$. So

$$\lim_{n\to\infty} \inf |(f^N)^n(x) - (f^N)^n(y)| = |x - y| > 0.$$

CASE 2: For x and y, one is periodic and the other not. Without loss of generality we assume $x \in P(f)$ and $y \notin P(f)$. There exists $k > 0$ such that x is a fixed point of f^k. Since $\mathrm{CR}(f) = \mathrm{CR}(f^k)$ and $P(f) = P(f^k)$, we have $y \in \mathrm{CR}(f^k) - P(f^k)$. Noting that $\mathrm{ent}(f) > 0$ iff $\mathrm{ent}(f^k) > 0$ (cf. [1] or [17]), it follows from Lemma 3.3 that $x \notin [a,b]$, where

$$a = \inf\{(f^k)^{2n}(y);\ n = 0, 1, 2, \ldots\},$$
$$b = \sup\{(f^k)^{2n}(y);\ n = 0, 1, 2, \ldots\}.$$

And hence $\inf\{|x - z|; z \in [a,b]\} > 0$. Thus we have for $N = 2k$

$$\begin{aligned}\lim_{n\to\infty} \inf |(f^N)^n(x) - (f^N)^n(y)| &= \lim_{n\to\infty} \inf |(f^{2k})^n(x) - (f^{2k})^n(y)| \\ &= \lim_{n\to\infty} \inf |x - (f^k)^{2n}(y)| \\ &\geq \inf\{|x - z|;\ z \in [a,b]\} > 0.\end{aligned}$$

CASE 3: $x \notin P(f)$ and $y \notin P(f)$. Since $[x,y] \cap P(f) \neq \emptyset$, there exists $p \in (x,y)$ such that $p \in P(f)$. Suppose the period of p is k. Then p is a fixed point of f^k. As stated in Case 2, $y \in \mathrm{CR}(f^k) - P(f^k)$ and $\mathrm{ent}(f^k) = 0$. Thus by Lemma 3.3 for

$$\alpha = \sup\{(f^k)^{2n}(x);\ n = 0,1,2,\ldots\} \quad \text{and} \quad \beta = \inf\{(f^k)^{2n}(y);\quad n = 0,1,2,\ldots\}$$

we have $\alpha < p < \beta$. One can check easily that $(*)$ holds for $N = 2k$.

The proof is complete.

LEMMA 3.5: *Let $x, y \in \mathrm{CR}(f)$. If $]f^n(x), f^n(y)[\bigcap P(f) = \emptyset$ for each $n = 0,1,2,\ldots$, then $\lim_{n\to\infty} |f^n(x) - f^n(y)| = 0$.*

Proof: For each $n \geq 0$, denote $I_n =]f^n(x), f^n(y)[$. We claim that $I_m \cap I_n = \emptyset$ whenever $m < n$.

To prove the claim, we restrict attention to the case

$$f^m(x) < f^n(x).$$

By Lemma 3.2, there is a fixed point e of f^{n-m} such that

$$f^m(x) < e < f^{n-m}(f^m(x)) = f^n(x).$$

Since $e \notin I_i$ for $i = m, n$, it follows that $I_m \subset [0, e)$ and $I_n \subset (e, 1]$, which implies $I_m \cap I_n = \emptyset$ and therefore the claim follows.

Let $L(I_n)$ denote the length of I_n. Then for each $N \geq 0$, by the claim,

$$\sum_{n=0}^{N} L(I_n) \leq 1.$$

This shows that the series $\sum_{n=0}^{\infty} L(I_n)$ converges and therefore

$$\lim_{n\to\infty} |f^n(x) - f^n(y)| = \lim_{n\to\infty} L(I_n) = 0.$$

Proof of Theorem C: If $x, y \in \mathrm{CR}(f)$ with $x \neq y$, then either for some $n \geq 0,]f^n(x), f^n(y)[\cap P(f) \neq \emptyset$, or for each $n \geq 0$, $]f^n(x), f^n(y)[\cap P(f) = \emptyset$. So, by Lemmas 3.4 and 3.5, any set containing both x and y is not chaotic under f.

Appendix

The aim of this appendix is to prove Lemma 2.3. The argument is patterned on that given in [21].

Let $S = \{0, 1, \ldots, k-1\}$ be a set of k symbols. We call A a symbol interval on S, if it is a finite sequence of elements of S. Suppose $A = (a_0 \ldots a_n)$ and $B = (b_0 \ldots b_m)$ are both symbol intervals on S. Define

$$AB = (a_0 \ldots a_n b_0 \ldots b_m), \tag{A*}$$

which is also a symbol interval on S. We say A occurs in B, and write $A \prec B$, if there exists $i \geq 0$ such that $a_j = b_{i+j}$ for $j = 0, \ldots, n$. Similarly, one may give the definition of a symbol interval occurring in a point $x \in \Sigma_k$.

Let K be a set of symbol intervals on S. By (A*), any finite sequence of elements of K is also a symbol interval on S, which will be called a **K-word**.

Let $\{I_i\}_{i=0}^{\infty}$ be a sequence of symbol intervals on S. For each $i \geq 0$, denote

$$K_i = \{a_0 I_0 a_1 I_1 \ldots I_{i-1} a_i;\ a_j \in S, 0 \leq j \leq i\}.$$

It is easy to see that K_i consists of k^{i+1} different symbol intervals on S. We call $\{I_i\}_{i=0}^{\infty}$ normal, if for each $i \geq 0$ the following follow:

(N1) There is at least one symbol interval occurring in I_i which contains successively all the k^{i+1} different elements of K_i.

(N2) $I_i a$ is a K_i-word for each $a \in S$.

LEMMA A.1: *Let $S = \{0, 1, \ldots, k-1\}$, $k \geq 2$. Then there is a normal sequence of symbol intervals on S.*

Proof: The proof will be given by induction.

Let $I_0 = (0\,1 \ldots k-1)$. It is easily seen that I_0 satisfies (N1) and (N2).

Suppose for $m \geq 1, \{I_0, I_1, \ldots, I_{m-1}\}$, satisfying (N1) and (N2), has been defined. Denote by J_m any symbol interval containing successively all the different symbol intervals in the set

$$K_m = \{a_0 I_0 \ldots a_{m-1} I_{m-1} a_m;\quad a_i \in S,\ 0 \leq i \leq m\}.$$

Put

$$I_m = J_m 0 I_0 0 I_1 \ldots 0 I_{m-1}.$$

It is not difficult to check that $\{I_0, I_1 \ldots, I_m\}$ satisfies (N1) and (N2). By induction, the lemma holds.

LEMMA A.2: *Let $\{I_i\}_{i=0}^{\infty}$ be a normal sequence of symbol intervals, $K_i = \{a_0 I_0 \dots a_{i-1} I_{i-1} a_i;\ \ a_j \in S,\ 0 \le j \le i\}$, $\forall i \ge 0$. For each $j \ge 0$, if $i \ge j$ and $A \in K_i$, then A is a K_j-word.*

Proof: For given $j \ge 0$, we use induction on i.

If $i = j$, then the conclusion clearly holds. Next suppose the conclusion holds for $i \le \ell$ and we prove that it also holds for $i = \ell + 1$. By the definition, there exist $a_0, \dots, a_{\ell+1} \in S$, such that

$$A = a_0 I_0 \dots a_\ell I_\ell a_{\ell+1}.$$

Set $B = a_0 I_0 \dots I_{\ell-1} a_\ell$. Then $B \in K_\ell$. By (N2), $I_\ell a_{\ell+1}$ is a K_ℓ-word, hence A is also. By the inductive assumption, each element of K_ℓ is a K_j-word and so A is a K_j-word. This proves that the conclusion holds for $i = \ell + 1$. We are done.

LEMMA A.3: *For $k \ge 2$, there exists in Σ_k an uncountable set E so that if $x = (x_0 x_1 \dots), y = (y_0 y_1 \dots)$ are different points in E, then $x_n \ne y_n$ for infinitely many n's.*

Proof: For any $x = (x_0 x_1 \dots), y = (y_0 y_1 \dots) \in \Sigma_k$, x is said to be equivalent to y, write $x \sim y$, if $x_n \ne y_n$ holds only for finitely many n's. It is easy to check that $\sim$ is an equivalence relation on Σ_k. Denote by $\Sigma_{k/\sim}$ the quotient space. We see that for each $x \in \Sigma_k$, the set $\{y \in \Sigma_k;\ y \sim x\}$ is countable. So $\Sigma_{k/\sim}$ is uncountable. Let E be an uncountable set formed by taking a representative from each equivalence class of $\Sigma_{k/\sim}$. Then E satisfies the requirements of the lemma.

LEMMA A.4: *Let $x = (x_0 x_1 \dots) \in \Sigma_k$, $k \ge 2$. If for any $j \ge 0$, there is an $N > 0$ such that $(x_0 \dots x_j)$ occurs in $(x_i x_{i+1} \dots x_{i+N})$ for each $i = 0, 1, 2, \dots$, then $x \in A(\sigma)$.*

The proof, being simple, is omitted.

Proof of Lemma 2.3: We restrict attention to the case $k = 2$. Put $S = \{0, 1\}$. For any symbol interval A on S, we denote by $L(A)$ the length of A, that is, the number of all the 0's and 1's in A.

Let $\{I_i\}_{i=0}^{\infty}$ be, as constructed in Lemma A.1, a normal sequence of symbol intervals on S. By Lemma A.3, we may choose an uncountable set E of Σ_2 such that if $x = (x_0 x_1 \dots), y = (y_0 y_1 \dots) \in E$ are different points, then $x_n \ne y_n$ for infinitely many n's. Define $\varphi\colon E \to \Sigma_2$ by $\varphi(x) = x_0I_0x_1I_1\ldots$ (to simplify the notation we write $x_0I_0x_1I_1\ldots$ rather than $(x_0I_0x_1I_1\ldots)$), $\forall x = (x_0x_1\ldots) \in E$. Put $D = \varphi(E)$. We now prove in succession:

(1) D is an uncountable chaotic set of σ.

It is easy to see that φ is injective. Thus, since E is uncountable, so also is D. For any $a \in D$, by the definition, there exists $(a_0a_1\ldots) \in E$ so that $a = a_0I_0a_1I_1\ldots$. Set

$$m_i = L(a_0I_0\ldots a_{i-1}I_{i-1}a_i).$$

Clearly

$$\sigma^{m_i}(a) = I_ia_{i+1}\ldots .$$

Observing that I_i does not depend on the selection of a and $L(I_i) \to \infty$ as $i \to \infty$, we have for any $x, y \in D$,

$$\text{(A}^{**}\text{)} \qquad \lim_{n\to\infty} \inf d(\sigma^n(x), \sigma^n(y)) \le \lim_{i\to\infty} d(\sigma^{m_i}(x),\ \sigma^{m_i}(y)) = 0$$

(see [1] for the metric d on Σ_2). Again by the property of E, for any $x, y \in D$ with $x \neq y$, there are infinitely many n's so that $x_n \neq y_n$. So we have

$$\lim_{n\to\infty} \sup d(\sigma^n(x), \sigma^n(y)) \ge 1.$$

And hence (1) holds.

(2) For each $y \in D$, $\omega(y, \sigma)$ is minimal and $D \subset \omega(y, \sigma)$.

Let $y = (y_0y_1\ldots) \in D$, where $y_i \in S$ for each $i \ge 0$. By the definition, there exists $b = (b_0b_1\ldots) \in E$ so that

$$y = \varphi(b) = b_0I_0b_1I_1\ldots = (y_0y_1\ldots).$$

Obviously,

$$(y_0y_1\ldots y_p) \prec b_0I_0b_1I_1\ldots I_{p-1}b_p, \quad \forall p \ge 0.$$

For given $p \ge 0$, by Lemma A.2, y may be viewed as an infinite sequence of symbol intervals of the form $a_0I_0\ldots a_pI_pa_{p+1}$. Set

$$N = 3L(b_0I_0\ldots b_pI_pb_{p+1}).$$

For given $i \ge 0$, it is easy to see that $(y_iy_{i+1}\ldots y_{i+N})$ contains a symbol interval of the form $a_0I_0\ldots a_pI_pa_{p+1}$, i.e.,

$$a_0I_0\ldots a_pI_pa_{p+1} \prec (y_iy_{i+1}\ldots y_{i+N}).$$

So, by (N1), we have

$$\begin{aligned}(y_0 y_1 \dots y_p) &\prec b_0 I_0 \dots b_{p-1} I_{p-1} b_p \\ &\prec I_p \\ &\prec a_0 I_0 \dots a_p I_p a_{p+1} \\ &\prec (y_i y_{i+1} \dots y_{i+N}).\end{aligned}$$

By Lemma A.4, $y \in A(\sigma)$. Again by Lemma 2.1, $\omega(y,\sigma)$ is a minimal set of σ.

Next we prove $D \subset \omega(y,\sigma)$. For any $z \in D$, by (A**),

$$\omega(y,\sigma) \cap \omega(z,\sigma) \neq \emptyset.$$

Since both are minimal, $\omega(y,\sigma) = \omega(z,\sigma)$. Thus

$$D \subset \bigcup_{z \in D} \omega(z,\sigma) = \omega(y,\sigma),$$

and then the proof of (2) is obtained.

Thus if $y \in D$, then $\omega(y,\sigma)$ is a minimal set containing the uncountable chaotic set D.

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